Published by Institute of Physics Publishing for SISSA

Received: *July 31, 2008*
Accepted: *September 3, 2008*
Published: *September 9, 2008*

Spectral curves, emergent geometry, and bubbling solutions for Wilson loops

Takuya Okuda

Kavli Institute for Theoretical Physics, University of California,
Santa Barbara, CA 93106, U.S.A.
E-mail: takuya@kitp.ucsb.edu

Diego Trancanelli

Physics Department, University of California,
Santa Barbara, CA 93106, U.S.A.
E-mail: dtrancan@physics.ucsb.edu

Abstract: We study the supersymmetric circular Wilson loops of $\mathcal{N} = 4$ super Yang-Mills in large representations of the gauge group. In particular, we obtain the spectral curves of the matrix model which captures the expectation value of the loops. These spectral curves are then proven to be precisely the hyperelliptic surfaces that characterize the bubbling solutions dual to the Wilson loops, thus yielding an example of a geometry emerging from an eigenvalue distribution. We finally discuss the Wilson loop expectation value from the matrix model and from supergravity.

Keywords: Matrix Models, AdS-CFT Correspondence, Strong Coupling Expansion.

Contents

1. Introduction

In gauge theory/gravity correspondences, a saddle point in the gauge theory path integral is expected to represent the space-time geometry in gravity. Since the saddle point is determined by the dynamics of the gauge theory, the space-time is said to be *emergent*. A notable example of such a phenomenon is the emergence of the sphere of the dual $AdS_5 \times S^5$ geometry from the matrix quantum mechanics governing the strong coupling dynamics of the constant modes of the scalars of $\mathcal{N} = 4$ super Yang-Mills compactified on a S^3 [1].[1]

When an operator is inserted in the gauge theory path integral, the saddle point, as well as the space-time represented by it, gets deformed. The new space-time develops bubbles of new cycles carrying fluxes. Such bubbling geometries were originally found for half-BPS local operators in $\mathcal{N} = 4$ super Yang-Mills theory in the context of the AdS/CFT correspondence [3, 4]. They were later generalized to include Wilson loops [5 − 7] and

[1] See [2] for a review of subsequent developments and a list of relevant references.

surface operators [8] of the $\mathcal{N}=4$ theory, while bubbling in topological string theory was found and studied in [9–11].

The current work revisits the bubbling geometries for circular supersymmetric Wilson loops in $\mathcal{N}=4$ super Yang-Mills. These geometries were constructed in a complete form in reference [7]. The ten-dimensional space-time is a warped product

$$ds^2 = f_1^2 ds^2_{AdS_2} + f_2^2 ds^2_{S^2} + f_4^2 ds^2_{S^4} + ds^2_\Sigma \tag{1.1}$$

of $AdS_2 \times S^2 \times S^4$ and a half-plane Σ. The radii f_1, f_2, f_4 and all other supergravity fields are functions on Σ given in terms of two holomorphic functions, $\mathcal{A}$ and $\mathcal{B}$. In fact, Σ is naturally identified with the lower half-plane in one sheet of a hyperelliptic surface, also denoted by Σ, and $\mathcal{A}$ and $\mathcal{B}$ are constructed geometrically. Thus the data $(\Sigma, \mathcal{A}, \mathcal{B})$ completely characterize the bubbling solution.

In this paper, we demonstrate that the deformed saddle points in gauge theory represent the bubbling geometries by making use of a matrix model. It was conjectured in [12, 13] that the Wilson loop expectation value is captured by the Gaussian matrix model with a loop operator insertion. The conjecture was recently proved in reference [14], where it was also shown that the matrix is the constant mode of a scalar field.[2] We show that the saddle point configuration of the matrix eigenvalues back-reacts to the operator insertion and the hyperelliptic surface Σ arises as the spectral curve in a generalized sense that we explain in detail.[3] We also find an interpretation of $\mathcal{A}$ and $\mathcal{B}$ in the matrix model.

Concretely, the circular supersymmetric Wilson loop is defined as

$$W_R = \mathrm{Tr}_R\, P \exp \oint (iA + \theta^i \phi^i ds). \tag{1.2}$$

Here A is the gauge field and ϕ^i are the six real scalars. The integral is along a circle in $\mathbb{R}^4$, θ^i is a constant unit vector in $\mathbb{R}^6$, and s is the parameter of the circle such that $||dx/ds|| = 1$. The trace is taken in an irreducible representation R of $\mathrm{U}(N)$ or $\mathrm{SU}(N)$. Such R is specified by a Young tableau, which is also denoted by the same symbol R. The dual bubbling geometry has small curvature when R has long edges and it is characterized by a genus g hyperelliptic surface Σ, where g is the number of blocks in R (see figure 1). The Wilson loop expectation value is given by the matrix integral

$$\langle W_R \rangle_{\mathrm{YM}} = \frac{1}{\mathcal{Z}} \int [dM] e^{-\frac{2N}{\lambda}\mathrm{Tr} M^2} \mathrm{Tr}_R e^M. \tag{1.3}$$

The $N \times N$ matrix M is hermitian and $\mathcal{Z}$ is the partition function. For representations R that give rise to smooth bubbling geometries, we solve the matrix model in the limit where N is infinite and the 't Hooft coupling $\lambda \equiv g^2_{\mathrm{YM}} N$ is large. As it turns out, $\mathcal{A}$ and $\mathcal{B}$ are simply related to the resolvent $\omega(z)$ and the spectral parameter z of the matrix model:

$$\mathcal{A} \propto \omega(z) - 2z\,, \qquad \mathcal{B} \propto z + \mathrm{const.} \tag{1.4}$$

[2] In [15] it was argued that the matrix model arises as a mirror of the topological A-model for the $AdS_5 \times S^5$ superstring [16].

[3] It was originally argued by Yamaguchi [5] that the eigenvalue distribution of the matrix model characterizes the bubbling geometry.

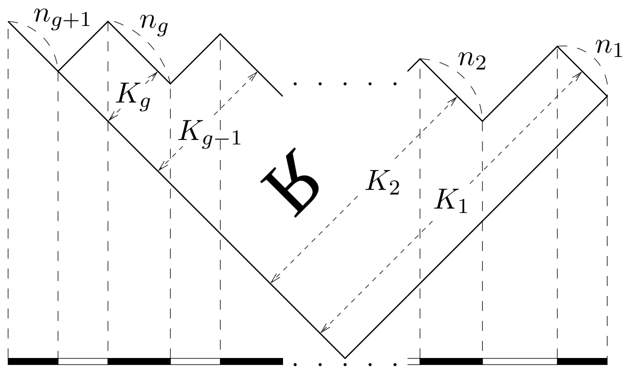

Figure 1: The Young tableau R is shown rotated and inverted. It consists of g blocks, the I-th one of them having n_I rows of length K_I. We set $K_{g+1} \equiv 0$ and $n_{g+1} \equiv N - \sum_{I=1}^{g} n_I$.

We also show that the resolvent is given by the indefinite integral of a meromorphic 1-form α on the same hyperelliptic surface Σ. The surface Σ is given by the equation

$$y^2 = H_{2g+2}(z)\,, \tag{1.5}$$

and the 1-form α by

$$\alpha = \partial\omega = \left(2 - 2\frac{a_{g+1}(z)}{\sqrt{H_{2g+2}(z)}}\right) dz\,. \tag{1.6}$$

The polynomials $H(z)$ and $a(z)$ have degrees $2g+2$ and $g+1$ respectively. We find from the matrix model analysis a set of constraints that determine the coefficients of $a(z)$ and $H(z)$ uniquely. These constraints are identical to the ones that arise in the bubbling geometry. The surface Σ is the spectral curve of the matrix model in the sense that the eigenvalue distribution is determined by Σ and α.

Given our large N solution of the matrix model, the Wilson loop expectation value can be easily computed. A natural question is whether it can also be reproduced in supergravity, by evaluating the on-shell action in the bubbling geometry background. We include in this paper some relevant calculations that will be useful for this purpose. In particular, we show that the on-shell supergravity Lagrangian is always a total derivative. This would imply that the on-shell action splits into two contributions, one coming from the new cycles of the bubbling geometry and the other given as a surface integral on the conformal boundary. It is the former contribution that we manage to compute exactly within an ansatz we make. This work does not address the latter contribution, which seems to require a holographic renormalization technology beyond the one currently available. Indeed, because the new cycles mix non-trivially the AdS_5 and S^5 directions, usual counter-terms in five-dimensional supergravity cannot be used, at least in a straightforward way.

It is however possible to use the identification of the matrix model and supergravity data to compare the correlators of the Wilson loop with local operators, namely chiral primaries and the energy-momentum tensor. This is reported in a companion paper [17].

JHEP09(2008)050

We structure the paper as follows. In section 2, we study the matrix model for Wilson loops dual to bubbling geometries. We solve the model, obtain its spectral curve, and show that it is the hyperelliptic surface that characterizes the bubbling geometry dual to the Wilson loop. Section 3 then focuses on the Wilson loop expectation value. Using our solution, we compute the Wilson loop expectation value for representations that correspond to smooth bubbling geometries. This reproduces the result of [18] in a certain limit. We next show that the on-shell supergravity Lagrangian is a total derivative and compute the contributions from the new cycles that appear in the bubbling geometry. We then conclude the paper by discussing the outlook in section 4. The appendices contain details used in the text.

2. Spectral curves and bubbling geometries

The expectation value of a circular Wilson loop in $\mathcal{N} = 4$ super Yang-Mills is captured by a Gaussian matrix model [12 – 14]. This was originally proposed for half-BPS loops in the fundamental representation (which are dual to fundamental strings in the bulk), but the conjecture has later been extended and checked to hold also for circular loops in arbitrary representations R of the gauge group [19 – 23] and for some loops preserving reduced amounts of supersymmetry [24 – 28]. The precise statement is that the Wilson loop expectation value for the $\mathrm{U}(N)$ theory is given by

$$\langle W_R \rangle_{\mathrm{U}(N)} = \frac{1}{\mathcal{Z}} \int [dM] \exp\left(-\frac{2N}{\lambda}\mathrm{Tr}M^2\right) \mathrm{Tr}_R e^M . \tag{2.1}$$

Here M is an hermitian matrix and the partition function $\mathcal{Z}$ of the matrix model is defined as the integral without the insertion $\mathrm{Tr}_R e^M$. We use the standard hermitian measure $[dM]$. In the absence of operator insertions, the eigenvalues are distributed in the large N limit according to the Wigner semi-circle law.[4]

To make better contact with the supergravity solution, it turns out to be more convenient to follow the procedure delineated in [11] and decompose M in $g+1$ sub-blocks $M^{(I)}$ of size $n_I \times n_I$. The expectation value of the loop is then given by several Gaussian matrix integrals correlated by interactions between the sub-blocks:

$$\langle W_R \rangle_{\mathrm{U}(N)} = \frac{1}{\mathcal{Z}} \int \prod_{I=1}^{g+1} [dM^{(I)}] e^{-\frac{2N}{\lambda}\sum_I \mathrm{Tr}(M^{(I)})^2} e^{K_I \mathrm{Tr} M^{(I)}} \prod_{I<J} \det \frac{(M^{(I)} \otimes 1 - 1 \otimes M^{(J)})^2}{1 - e^{-M^{(I)}} \otimes e^{M^{(J)}}} . \tag{2.2}$$

The eigenvalues of $M^{(I)}$ for fixed I are distributed along some interval $[e_{2I}, e_{2I-1}]$. In the following, we drop the exponential interactions by replacing $(1 - e^{-M^{(I)}} \otimes e^{M^{(J)}})$ with 1. This is a consistent approximation in the limit

$$\lambda \gg 1 , \qquad g^2_{\mathrm{YM}} n_I = \mathcal{O}(\lambda) , \qquad g^2_{\mathrm{YM}}(K_I - K_{I+1}) = \mathcal{O}(\lambda^{1/2}) , \tag{2.3}$$

[4] Pedagogical references on general matrix models include [29, 30].

because $e_{2I-1} - e_{2I} = \mathcal{O}(\sqrt{g_{\rm YM}^2 n_I})$ and $e_{2I} - e_{2I+1} = \mathcal{O}(g_{\rm YM}^2(K_I - K_{I+1}))$ as one can see from the saddle point equations below.

Going to the eigenvalue basis, the matrix model in (2.2) becomes (here $i = 1, \dots, n_I$ labels the eigenvalues of the I-th sub-block)

$$\langle W_R \rangle_{{\rm U}(N)} \propto \int \prod_{I,i} dm_i^{(I)} \exp\left[-\frac{2N}{\lambda} \left(m_i^{(I)}\right)^2 + K_I m_i^{(I)} \right] \prod_{(I,i)<(J,j)} \left[m_i^{(I)} - m_j^{(J)} \right]^2 . \tag{2.4}$$

We have introduced a linear ordering in the set of all the eigenvalues so that the last product is taken over distinct pairs of eigenvalues. The saddle point equations are

$$-\frac{4N}{\lambda} m_i^{(I)} + K_I + 2 \sum_{(J,j)\neq(I,i)} \frac{1}{m_i^{(I)} - m_j^{(J)}} = 0. \tag{2.5}$$

By defining the resolvent

$$\omega(z) \equiv g_{\rm YM}^2 \sum_{(I,i)} \frac{1}{z - m_i^{(I)}}, \tag{2.6}$$

the eqs. (2.5) can be written, for $x \in [e_{2I}, e_{2I-1}]$, as

$$-4x + g_{\rm YM}^2 K_I + \omega_+(x) + \omega_-(x) = 0\,, \tag{2.7}$$

where $\omega_\pm(x) \equiv \omega(x \pm i\epsilon)$.

2.1 A hyperelliptic surface as the spectral curve

By differentiating eq. (2.7), one can see that $\omega'_\pm = 4 - \omega'_\mp$, so that the combination

$$\omega'(4 - \omega') \tag{2.8}$$

is invariant when crossing the cut. Let us now consider the behavior of this expression close to a branch point, say e_1. The eigenvalues are expected to produce square root branch cuts. Since $\omega(z)$ satisfies eq. (2.7), locally it is given by

$$\omega \sim 2z - \frac{1}{2} g_{\rm YM}^2 K_1 + c\sqrt{z - e_1}, \tag{2.9}$$

where c is some constant. Then

$$\omega'(4 - \omega') \sim \left(2 + \frac{c}{2\sqrt{z-e_1}}\right)\left(2 - \frac{c}{2\sqrt{z-e_1}}\right) = 4 - \frac{c^2}{4(z-e_1)}. \tag{2.10}$$

The same behavior is found for every branch point e_i:

$$\omega'(4 - \omega') \sim \frac{C_i}{z - e_i} \quad \text{as} \quad z \to e_i\,, \qquad C_i = \text{const.}, \tag{2.11}$$

so the combination

$$\omega'(4-\omega') - \sum_{i=1}^{2g+2} \frac{C_i}{z-e_i} \tag{2.12}$$

is regular everywhere on the complex plane. The first term behaves as $\mathcal{O}(1/z^2)$ for large z by the definition of ω. Thus the combination must vanish everywhere and, in addition, the second term has to be of the form

$$-\sum_{i=1}^{2g+2} \frac{C_i}{z-e_i} = -\frac{f_{2g}(z)}{H_{2g+2}(z)}\,, \tag{2.13}$$

with $f_{2g}(z)$ a polynomial of degree $2g$ and

$$H_{2g+2}(z) \equiv \prod_{i=1}^{2g+2} (z-e_i)\,. \tag{2.14}$$

The solution to the quadratic equation

$$\omega'(4-\omega') = \frac{f_{2g}(z)}{H_{2g+2}(z)} \tag{2.15}$$

is then

$$\omega' = 2 - \sqrt{4 - \frac{f_{2g}(z)}{H_{2g+2}(z)}} \equiv 2 - 2\frac{a_{g+1}(z)}{\sqrt{H_{2g+2}(z)}}. \tag{2.16}$$

Here we have selected the negative sign in front of the square root to guarantee the correct behavior for $z \to \infty$. In introducing the monic polynomial $a_{g+1}(z) = z^{g+1} + \cdots$, we noted that $H_{2g+2} - f_{2g}/4$ has to be a perfect square, so that the only singularities of ω' are the branch points e_i.

We can geometrically interpret eq. (2.16) by saying that the resolvent $\omega(z)$ is the indefinite integral

$$\omega(z) = \int_\infty^z \alpha \tag{2.17}$$

of a meromorphic 1-form

$$\alpha = \left(2 - 2\frac{a_{g+1}(z)}{\sqrt{H_{2g+2}(z)}}\right) dz \tag{2.18}$$

on the hyperelliptic curve defined by

$$y^2 = H_{2g+2}(z)\,. \tag{2.19}$$

The only singularity of the 1-form α is the double pole at $z=\infty$ on the second sheet.

JHEP09(2008)050

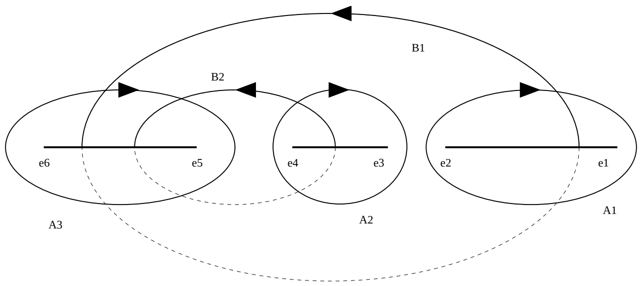

Figure 2: The A- and B-cycles of the hyperelliptic surface Σ of genus $g = 2$.

2.2 Parameters and constraints

The parameters in the definition of the spectral curve and the one-form are the $3g + 3$ coefficients of the two monic polynomials $a_{g+1} \equiv a$ and $H_{2g+2} \equiv H$. Let us study the constraints that determine these parameters.

The constraints are most concisely expressed in terms of period integrals, so let us introduce the A- and B-cycles of the hyperelliptic surface in the standard way (see figure 2): the cycle A_I (with $I = 1, \ldots, g + 1$) circles the I-th cut $[e_{2I}, e_{2I-1}]$ clockwise. Only the first g of the A-cycles are independent, since $A_{g+1} = -A_1 - \cdots - A_g$. The cycle B_I (with $I = 1, \ldots, g$) goes through the I-th and the $(g + 1)$-th cuts and has intersection numbers $\#(A_I \cap B_J) = \delta_{IJ}$ for $J = 1, 2, \ldots, g$.

1. The first $g+1$ constraints come from the requirement that the resolvent $\omega(z)$ should be single-valued on the physical sheet. Since it is obtained by integrating the one-form α, we need that

$$\oint_{A_I} \alpha = 0, \qquad I = 1, \ldots, g + 1. \tag{2.20}$$

These $g + 1$ constraints are all independent: even though $\sum_{I=1}^{g+1} A_I$ is a trivial cycle in homology, the condition $\int_{\sum A_I} \alpha = 0$ applied to (2.18) is non-trivial and ensures that no logarithmic term appears in the expansion of ω around $z = \infty$.

2. According to the saddle point equations (2.7), the value of ω along the cycle B_I goes from ω to $4z - \omega$ in passing through the $(g + 1)$-th cut from the first to the second sheet (recall that $K_{g+1} = 0$), and then from $4z - \omega$ to $\omega + g_{\rm YM}^2 K_I$ when coming back to the first sheet across the I-th cut. In terms of the one-form α, we get g conditions

$$\oint_{B_I} \alpha = g_{\rm YM}^2 K_I\,, \qquad I = 1, \ldots, g. \tag{2.21}$$

3. Since the I-th cut contains n_I eigenvalues, the definition (2.6) implies the following $g + 1$ conditions

$$\oint_{A_I} \omega\, dz = -2\pi i g_{\rm YM}^2 n_I\,, \qquad I = 1, \ldots, g + 1. \tag{2.22}$$

The integral should be performed on the first sheet.

4. The $3g+2$ conditions above determine $a_{g+1}(z)$ and $H_{2g+2}(z)$ up to a shift of z. The last condition that fixes this ambiguity is

$$\omega(e_{2g+2}) = 2e_{2g+2}\,, \tag{2.23}$$

which follows form (2.7) recalling that $K_{g+1} = 0$.

We check now that $\omega(z)$ given by (2.17) together with the constraints (2.20)–(2.23) automatically satisfies the saddle point equations (2.7). For this we need to evaluate ω just above and below each branch cut $[e_{2I}, e_{2I-1}]$. Since we know the value of ω at $z = e_{2g+2}$, we only need to integrate α from e_{2g+2} to e_{2I} along an arbitrary path on the first sheet, and then from e_{2I} to $x \pm i\epsilon$ with $x \in [e_{2I}, e_{2I-1}]$ along the cut. The key points are that

$$4\int_{e_{2g+2}}^{e_{2I}} \frac{a(z)}{\sqrt{H(z)}}dz = g_{\rm YM}^2 K_I\,, \tag{2.24}$$

as follows from the condition (2.21), and that

$$\sqrt{H(x+i\epsilon)} = -\sqrt{H(x-i\epsilon)} \tag{2.25}$$

on the cut. For $x \in [e_{2I}, e_{2I-1}]$ we have

$$\begin{aligned}
\omega_+(x)+\omega_-(x) &= 2\omega(e_{2g+2}) + 2\int_{e_{2g+2}}^{e_{2I}}\left(2-2\frac{a}{\sqrt{H}}\right)dz \\
&\quad + \int_{[e_{2I},x]+i\epsilon}\left(2-2\frac{a(x')}{\sqrt{H(x')}}\right)dx' + \int_{[e_{2I},x]-i\epsilon}\left(2-2\frac{a(x')}{\sqrt{H(x')}}\right)dx' \\
&= 4e_{2g+2} + 4(e_{2I}-e_{2g+2}) - g_{\rm YM}^2 K_I + 4(x-e_{2I}) \\
&= 4x - g_{\rm YM}^2 K_I\,,
\end{aligned} \tag{2.26}$$

so we see that the saddle point equations (2.7) are satisfied. Thus at this point we have found the exact solution of the matrix model (2.4) in the large N limit.

2.3 Comparison

What remains to be shown is that the spectral curve (2.19) is the hyperelliptic surface that appears as part of the bubbling solution for a Wilson loop [7].

The bubbling geometry is a warped product of $AdS_2 \times S^2 \times S^4$ and a half-plane, as we have mentioned in the introduction. This half-plane is taken to be the lower half-plane in one sheet of the hyperelliptic surface given by

$$s^2 = \prod_{i=1}^{2g+1}(u-\tilde{e}_i). \tag{2.27}$$

The branch points of the surface are at $u = \tilde{e}_i$ (with $i = 1, \ldots, 2g+1$) and $u = \tilde{e}_0 \equiv \tilde{e}_{2g+2} \equiv \infty$. (Notation changed from [7]: $e_i^{\rm there} = \tilde{e}_i^{\rm here}$.) The constant u_0 and the branch points $\tilde{e}_i$ are all real and ordered as follows:

$$\tilde{e}_{2g+1} < \tilde{e}_{2g} < \ldots < \tilde{e}_1 < u_0. \tag{2.28}$$

Though the r.h.s. of (2.27) is a polynomial of degree $2g+1$ instead of $2g+2$, the equation can be transformed to the form (2.19) by a Möbius transformation on u.

All the supergravity fields are expressed in terms of two holomorphic functions $\mathcal{A}$ and $\mathcal{B}$ on Σ given by

$$\partial\mathcal{A} = -i\frac{P(u)du}{(u-u_0)^2 s(u)}, \tag{2.29}$$

$$\mathcal{B} = -i\frac{1}{u-u_0}. \tag{2.30}$$

The polynomial $P(u)$ has real coefficients and is of degree $g+1$. The real part of $\mathcal{A}$ must vanish on $[\tilde{e}_{2I+1}, \tilde{e}_{2I}]$ to ensure regularity of the solution, so there are constraints

$$\int_{[\tilde{e}_{2I},\tilde{e}_{2I-1}]-i\epsilon} \partial\mathcal{A} = 0, \qquad I = 1,\ldots,g+1. \tag{2.31}$$

The branch cuts $[\tilde{e}_{2I-1}, \tilde{e}_{2I-2}]$ represent three-cycles of topology S^3 that arise from the geometric transition of D5-branes, so they carry RR three-form fluxes. Since each column in the Young tableau R represents a D5-brane [20, 31], the flux carried by the I-th cycle is proportional to $K_I - K_{I+1}$, the number of columns in the I-th block:

$$8\pi i \int_{[\tilde{e}_{2I-1},\tilde{e}_{2I-2}]} \partial\mathcal{A} + c.c. = \int_{S^3} F^{\mathrm{RR}}_{(3)} = 4\pi^2(K_I - K_{I+1})\alpha' \tag{2.32}$$

for $I = 1,\ldots,g$. Similarly, the segment $[\tilde{e}_{2I}, \tilde{e}_{2I-1}]$ represents a five-cycle of topology S^5 that arises from the geometric transition of n_I D3-branes [19, 31] and carries RR five-form flux. As we show in appendix B

$$8\pi^2 i \int_{[\tilde{e}_{2I},\tilde{e}_{2I-1}]-i\epsilon} (\mathcal{A}\partial\mathcal{B} - \mathcal{B}\partial\mathcal{A}) + c.c. = \int_{S^5} F_{(5)} = 4\pi^4\alpha'^2 n_I \tag{2.33}$$

for $I = 1,\ldots,g+1$.

Shifting the imaginary part of $\mathcal{A}$ does not affect the physical fields. It is natural to fix this ambiguity by requiring that

$$\lim_{u\to\infty} \mathcal{A} = 0. \tag{2.34}$$

The constraints (2.31)–(2.34) are equivalent to (2.20)–(2.23), respectively, if we make the identification

$$\omega - 2z = i\frac{8}{\alpha'}g_s\mathcal{A}, \qquad \mathcal{B} = i\frac{\alpha'}{4}(z - e_{2g+2}). \tag{2.35}$$

Equivalently, we have

$$\mathcal{A} = i\frac{\alpha'}{4g_s}\int_{e_{2g+2}}^{z} \frac{a(z')}{\sqrt{H(z')}}dz', \qquad u - u_0 = \frac{4}{\alpha'}\frac{1}{e_{2g+2} - z}. \tag{2.36}$$

Note that $g_{\mathrm{YM}}^2 = 4\pi g_s$. Thus we have showed that the spectral curve of the matrix model is precisely the hyperelliptic surface that characterizes the bubbling geometry.

JHEP09(2008)050

2.4 SU(N) gauge group

So far we have focused on the U(N) gauge group case. It is easy to describe what changes for a SU(N) theory. First, the Wilson loop expectation value of the gauge theory is related to the matrix model by a simple modification of (2.1):

$$\langle W_R \rangle_{\mathrm{SU}(N)} = \frac{1}{\mathcal{Z}} \int [dM] \exp\left(-\frac{2N}{\lambda} \mathrm{Tr} M^2 \right) \mathrm{Tr}_R e^{M'}, \tag{2.37}$$

where

$$M' = M - \frac{1}{N} (\mathrm{Tr} M) 1_{N \times N} \tag{2.38}$$

is the traceless part of M. Since

$$\mathrm{Tr}_R e^{M'} = e^{-\frac{|R|}{N} \mathrm{Tr} M} \mathrm{Tr}_R e^M, \tag{2.39}$$

the saddle point equation (2.7) for the I-th cut becomes

$$-4x + g_{\mathrm{YM}}^2 (K_I - |R|/N) + \omega_+(x) + \omega_-(x) = 0. \tag{2.40}$$

Therefore the resolvents of the U(N) and SU(N) theories are simply related by a shift of the argument:

$$\omega_{\mathrm{SU}(N)}(z) = \omega_{\mathrm{U}(N)}(z + |R|/4N). \tag{2.41}$$

Equivalently, the eigenvalue distribution is simply shifted by a constant so that the average position of the eigenvalues is the origin. The relations between ω, z and $\mathcal{A}$, $\mathcal{B}$ become

$$\omega - 2(z + |R|/4N) = i \frac{8}{\alpha'} g_s \mathcal{A}, \qquad \mathcal{B} = i \frac{\alpha'}{4} (z - e_{2g+2}), \tag{2.42}$$

where $e_{2g+2} \equiv e_{2g+2}^{\mathrm{U}(N)} + |R|/4N$ is the last branch point in the SU(N) case.

3. Wilson loop expectation value

Given our identification of the matrix model and supergravity data, it is natural to compare various physical quantities computed on both sides. A companion paper [17] studies the correlators of Wilson loops with local operators, such as chiral primaries and the energy-momentum tensor, finding agreement between gauge theory and supergravity analysis. Another natural quantity to compare is the Wilson loop expectation value, which we study in this section. On the Yang-Mills side, we compute it using our large N solution of the matrix model. We also discuss the supergravity computation though we do not complete it in this paper.[5] First we prove that the on-shell supergravity Lagrangian is always a total derivative. Then we show that the action contains contributions from the new cycles of the bubbling geometry and also from the boundary of space-time. We compute the first kind of contributions. Issues with the second type are discussed in section 4.

[5] The computation of the expectation value of a loop dual to D3 and D5 branes [31, 32] has been performed in [19 – 22], both using the matrix model and the DBI action.

JHEP09(2008)050

3.1 Wilson loop expectation value from the matrix model

To the leading order in the saddle point approximation, the normalized Wilson loop expectation value is given by

$$\langle W_R \rangle = e^{-(\mathcal{S}_{\text{mat}} - \mathcal{S}_0)}, \tag{3.1}$$

where $\mathcal{S}_{\text{mat}}$ and $\mathcal{S}_0$ are the on-shell actions of the Gaussian matrix model with and without Wilson loop insertion. We now proceed with computing these actions.

Again, we begin with the case of a $\mathrm{U}(N)$ gauge group. The on-shell value of the matrix model action is given by

$$\begin{aligned} -\mathcal{S}_{\text{mat}} &= \sum_{I,i} \left[-\frac{2N}{\lambda} \left(m_i^{(I)} \right)^2 + K_I m_i^{(I)} \right] + \sum_{(I,i)<(J,j)} \log \left[m_i^{(I)} - m_j^{(J)} \right]^2 \\ &= N \sum_I \int_{[e_{2I}, e_{2I-1}]} dx \rho(x) \left[-\frac{2N}{\lambda} x^2 + K_I x \right] + N^2 \int_{\mathbb{R}} dx dy \rho(x) \rho(y) \log |x-y|, \end{aligned} \tag{3.2}$$

where the eigenvalue density

$$\rho(x) = \frac{1}{N} \sum_{I,i} \delta(x - m_i^{(I)}) \tag{3.3}$$

is related to the resolvent by

$$\rho(x) = \frac{i}{2\pi\lambda} \left(\omega_+(x) - \omega_-(x) \right), \qquad \omega(z) = \lambda \int_{\mathbb{R}} dx \frac{\rho(x)}{z - x}. \tag{3.4}$$

In the limit in which the cuts are well separated, the last term in (3.2) can be dropped, and by using the eigenvalue density $\rho(x)$ given by

$$\sum_I \frac{n_I}{N} \delta\left(x - g_{\text{YM}}^2 K_I / 4 \right), \tag{3.5}$$

we easily reproduce the results of [18].

The expression (3.2) may be enough for comparison with supergravity although we do not see how the double integral can appear in gravity. We now rewrite (3.2) in a form that involves no double integral. First, let us use the density and a principal value integral to re-express (2.5):

$$-4x + g_{\text{YM}}^2 K_I + 2\lambda \mathrm{P} \int_{\mathbb{R}} dy \rho(y) \frac{1}{x-y} = 0 \quad \text{for} \quad x \in [e_{2I-1}, e_{2I}]. \tag{3.6}$$

This equation can be integrated to yield

$$-2x^2 + g_{\text{YM}}^2 K_I x + 2\lambda \int_{\mathbb{R}} dy \rho(y) \log|x-y| = 2 g_{\text{YM}}^2 c_I \quad \text{for} \quad x \in [e_{2I-1}, e_{2I}], \tag{3.7}$$

where c_I is an integration constant. The on-shell matrix action is then

$$-\mathcal{S}_{\text{mat}} = N \sum_{I=1}^{g+1} \int_{[e_{2I}, e_{2I-1}]} dx \rho(x) \left[-\frac{N}{\lambda} x^2 + \frac{1}{2} K_I x + c_I \right]. \tag{3.8}$$

One expression for the Wilson loop expectation value that does not involve a double integral or c_I is obtained by using (3.7) with $x = e_{2I-1}$ and $x = e_{2I}$:

$$\log\langle W_R\rangle_{\mathrm{U}(N)} = N\sum_{I=1}^{g+1}\int_{[e_{2I},e_{2I-1}]} dx\rho(x)\Bigg[-\frac{N}{\lambda}x^2+\frac{1}{2}K_I x-\frac{N}{2\lambda}(e_{2I-1}^2+e_{2I}^2)+\frac{K_I(e_{2I-1}+e_{2I})}{4} + \frac{1}{2}\sum_J n_J \log(e_{2J-1}-x)(x-e_{2J})\Bigg] - \log\sqrt{\lambda}+3/4+\log 2\,, \qquad (3.9)$$

where we used

$$\mathcal{S}_0 = N^2\left(-\log\sqrt{\lambda}+3/4+\log 2\right) \qquad (3.10)$$

that follows from the density $\rho_0(x) = (1/\pi\lambda)\sqrt{\lambda - x^2}$ for Wigner's distribution.

For the SU(N) theory, we simply replace K_I by $K_I - |R|/N$:

$$\log\langle W_R\rangle_{\mathrm{SU}(N)} = N\sum_{I=1}^{g+1}\int_{[e_{2I},e_{2I-1}]} dx\rho(x)\Bigg[-\frac{N}{\lambda}x^2+\frac{1}{2}K_I x-\frac{N}{2\lambda}(e_{2I-1}^2+e_{2I}^2) + \frac{(K_I-|R|/N)(e_{2I-1}+e_{2I})}{4}+\frac{1}{2}\sum_J n_J\log(e_{2J-1}-x)(x-e_{2J})\Bigg] - \log\sqrt{\lambda}+3/4+\log 2. \qquad (3.11)$$

In this formula $\rho(x)$ and e_i are the density and the branch points in the SU(N) case, and we have used the fact that the average eigenvalue vanishes to remove a shift of K_I in the second term inside the bracket.

3.2 Wilson loop expectation value from supergravity

Let us now turn to supergravity. The solution in [7] is for an infinite straight line along the Lorentzian time, whereas the matrix model model computation is appropriate for a circle in Euclidean signature. This is not a problem, since both the straight line and the circle preserve the same isometry $\mathrm{SO}(2,1)\times\mathrm{SO}(3)\times\mathrm{SO}(5)$ (albeit realized differently in the two cases). We can then extend the solution of [7] to the circular case via a Wick rotation, considering a fibration with the Euclidean factor $\mathbb{H}_2$, rather than AdS_2. This difference will not play any significant role in our analysis, so that we shall consider for simplicity the Lorentzian signature. The Wilson loop expectation value is then given by $\langle W_R\rangle = \exp(-\mathcal{S}_E)$ after the Wick rotation that identifies $-\mathcal{S}_E$ with $i\mathcal{S}_L$, where $\mathcal{S}_E$ and $\mathcal{S}_L$ are the Euclidean and Lorentzian on-shell actions.

3.2.1 The on-shell lagrangian is a total derivative

We begin our discussion of the supergravity action by showing that the on-shell Lagrangian density always has to be a total derivative, if it is a homogeneous function of the fields of non-zero degree. It seems well-known that the supergravity Lagrangian is a total derivative if the equations of motion are satisfied, though we do not know a reference that makes the general statement explicitly.

JHEP09(2008)050

The argument is simple. Suppose the Lagrangian $\mathcal{L}(\phi)$ depends on the fields ϕ^i and their derivatives. There can be second or higher derivatives. When we take the variation of $\mathcal{L}$ with respect to arbitrary changes $\delta\phi^i$, in general we get terms that contain derivatives of $\delta\phi^i$. By definition, the equations of motion $\mathcal{E}_i(\phi) = 0$ are obtained by rewriting $\delta\mathcal{L}$ as

$$\delta\mathcal{L} = \sum_i \delta\phi^i \mathcal{E}_i(\phi) + \sum_i \mathcal{D}_i(\delta\phi^i; \phi), \tag{3.12}$$

where $\mathcal{D}_i$ is the total derivative term that is linear in $\delta\phi^i$. If the Lagrangian is homogeneous, there are (usually integers) numbers $n_{\mathcal{L}}$ and n_i such that

$$\mathcal{L}(\Omega^{n_i}\phi^i) = \Omega^{n_{\mathcal{L}}}\mathcal{L}(\phi^i) \tag{3.13}$$

for any constant Ω. We call n_i the dimensions of the fields. By choosing $\Omega = 1 + \epsilon$ so that $\delta\phi^i = \epsilon\, n_i \phi^i$, we find that

$$\epsilon\, n_{\mathcal{L}} \mathcal{L}(\phi) = \sum_i \epsilon\, n_i \phi^i \mathcal{E}_i(\phi) + \sum_i \mathcal{D}_i(\epsilon\, n_i \phi^i; \phi). \tag{3.14}$$

If the equations of motion are satisfied, the Lagrangian is a total derivative:

$$\mathcal{L}(\phi) = \sum_i \frac{n_i}{n_{\mathcal{L}}} \mathcal{D}_i(\phi^i; \phi). \tag{3.15}$$

We now apply the above consideration to the type IIB supergravity action[6]

$$\begin{aligned} 2\kappa^2\, \mathcal{S} = & \int d^{10}x\, \sqrt{-g}\left(R - \frac{1}{2}\frac{\partial_M \tau \partial^M \bar{\tau}}{(\mathrm{Im}\,\tau)^2}\right) \\ & + \int \left(-\frac{1}{2} M_{ab} H^a_{(3)} \wedge \star H^b_{(3)} - 4F_{(5)} \wedge \star F_{(5)} - \epsilon_{ab} C_{(4)} \wedge H^a_{(3)} \wedge H^b_{(3)}\right). \end{aligned} \tag{3.16}$$

The action is written essentially in the convention of [34] and contains various combinations of the fields:

$$\begin{aligned} &\tau = C_{(0)} + ie^{-\varphi}\,, \qquad\qquad (M_{ab}) = \mathrm{diag}(e^{-\varphi}, e^{\varphi})\,, \\ &F_{(5)} = dC_{(4)} + \frac{1}{8}\epsilon_{ab} B^a_{(2)} \wedge dB^b_{(2)}, \end{aligned} \tag{3.17}$$

where $H^a_{(3)} = dB^a_{(2)}$ and $a =$ NS, RR. First note that the action is homogeneous of degree 8 if we assign dimension 2 to the metric g_{MN} and p to all p-form fields (scalars are zero-forms). So our argument applies. Since the scalars have vanishing dimensions, we can ignore their variations. Then under arbitrary variations of the fields except the scalars, the action changes as

$$\begin{aligned} 2\kappa^2 \delta\mathcal{S} = & \int d^{10}x \sqrt{-g} \nabla^M (\nabla^N \delta g_{MN} - g^{PQ} \nabla_M \delta g_{PQ}) \\ & + \int d\Big(- M_{ab} \delta B^a_{(2)} \wedge \star H^b_{(3)} - 2\epsilon_{ab} C_{(4)} \wedge \delta B^a_{(2)} \wedge H^b_{(3)} \\ & \qquad - 8\delta C_{(4)} \wedge \star F_{(5)} + \epsilon_{ab} \delta B^a_{(2)} \wedge B^b_{(2)} \wedge \star F_{(5)}\Big) \end{aligned} \tag{3.18}$$

[6] Self-duality of the five-form, $F_{(5)} = \star F_{(5)}$, does not follow from this action, but has to be imposed by hand. One can consider other actions where self-duality is implemented with an auxiliary field. In the case [33] we looked at, the on-shell value does not seem to change.

JHEP09(2008)050

up to terms that vanish on-shell. By setting

$$\delta g_{MN} = 2\epsilon g_{MN}, \quad \delta B^a_{(2)} = 2\epsilon B^a_{(2)}, \quad \delta C_{(4)} = 4\epsilon C_{(4)} \tag{3.19}$$

and using $\delta\mathcal{S} = 8\epsilon\mathcal{S}$, we conclude that the on-shell action is given by

$$2\kappa^2\mathcal{S} = \int d\left(-\frac{1}{4}M_{ab}B^a_{(2)} \wedge \star H^b_{(3)} - \frac{1}{2}\epsilon_{ab}C_{(4)} \wedge B^a_{(2)} \wedge H^b_{(3)} - 4C_{(4)} \wedge \star F_{(5)}\right). \tag{3.20}$$

We thus see that we only need the two- and four-form fields to compute this part of the action. Note that so far we have not committed to any particular solution.

3.2.2 Contributions from new cycles

In the solution of [7], the NS two-form is along the AdS_2 directions while the RR two-form along the S^2 directions. The RR four-form has two components, one in the $AdS_2 \times S^2$ and the other in the S^4 directions. One has then

$$B^{\rm NS}_{(2)} = b_1\hat{e}^{01}\,, \quad B^{\rm RR}_{(2)} = b_2\hat{e}^{23}\,, \quad C_{(4)} = -j_1\hat{e}^{0123} + j_2\hat{e}^{4567}, \tag{3.21}$$

where $\hat{e}^{01}$, $\hat{e}^{23}$, and $\hat{e}^{4567}$ are the volume forms of AdS_2, S^2, and S^4, respectively, all normalized to unit radius. Note that b_1, b_2, j_1, and j_2 are real functions on Σ. Recall now that the S^2 and S^4 radii vanish on segments of the real axis of Σ. Thus $\hat{e}^{23}$ and $\hat{e}^{4567}$ are not globally defined forms in the ten-dimensional space-time, while $\hat{e}^{01}$ is. This implies that the Chern-Simons term in (3.16) is not globally defined. We can make it globally defined by adding further total derivative terms

$$2\kappa^2\mathcal{S}_1 = \int d\left(2C_{(4)} \wedge B^{\rm NS}_{(2)} \wedge H^{\rm RR}_{(3)} - \frac{1}{16}B^{\rm NS}_{(2)} \wedge B^{\rm NS}_{(2)} \wedge d\left(B^{\rm RR}_{(2)} \wedge B^{\rm RR}_{(2)}\right)\right) \tag{3.22}$$

so that the new Chern-Simons term in $2\kappa^2(\mathcal{S} + \mathcal{S}_1) \equiv 2\kappa^2\mathcal{S}_{\rm bulk}$ is

$$\int 2F_{(5)} \wedge B^{\rm NS}_{(2)} \wedge H^{\rm RR}_{(3)}. \tag{3.23}$$

The on-shell action is then given by

$$2\kappa^2\mathcal{S}_{\rm bulk} = \int d\left(-\frac{1}{4}M_{ab}B^a_{(2)} \wedge \star H^b_{(3)} - B^{\rm NS}_{(2)} \wedge B^{\rm RR}_{(2)} \wedge dC_{(4)}\right), \tag{3.24}$$

where we took into account (3.21).

Since some of the forms in (3.24) are not globally defined, we need caution in applying the Poincaré lemma. Some terms in (3.24) are contributions of the non-trivial cycles in the bubbling geometry, while the rest are from the boundary of space-time. We focus on the former contributions. The latter should be combined with counter-terms we do not discuss in the present work.

With our ansatz, the Hodge duals of the three-forms are given by

$$\star F^{\rm NS}_{(3)} = \frac{f_2^2 f_4^4}{f_1^2} \star db_1 \wedge \hat{e}^{234567}, \tag{3.25}$$

$$\star F^{\rm RR}_{(3)} = \frac{f_1^2 f_4^4}{f_2^2} \star db_2 \wedge \hat{e}^{014567}. \tag{3.26}$$

We have by the Poincaré lemma

$$2\kappa^2 \mathcal{S}_{\rm bulk} = V \int_{\partial\Sigma} \left(-\frac{1}{4} e^{-\varphi} \frac{f_2^2 f_4^4}{f_1^2} b_1 \star db_1 - \frac{1}{4} e^{\varphi} \frac{f_1^2 f_4^4}{f_2^2} b_2 \star db_2 - b_1 b_2 dj_2 \right). \qquad (3.27)$$

By $\partial\Sigma$ we denote the real axis as well as a large semi-circle on the lower half-plane. We cannot meaningfully separate contributions from the two components of $\partial\Sigma$ because adding an exact form in the integrand of (3.27) mixes them. In (3.27) we have made the important assumption that the volume of AdS_2 is regularized in a way independent of the position on $\partial\Sigma$. We have denoted the volume of $AdS_2 \times S^2 \times S^4$ by V. In a more complete calculation of the on-shell action, this assumption may need to be modified.

In appendix C, we study how various quantities in (3.27) behave in the asymptotic region $z \to \infty$. If we choose the coordinate to be the spectral parameter z in the SU(N) case, both b_1 and b_2 vanish as $z \to \infty$ while j_2 remains finite. Thus the contribution from the semi-circle in this parametrization vanishes.

On the real axis, the first term in (3.27) never contributes because it contains positive powers of radii of the two spheres and always vanishes. The remaining two terms nicely combine to give

$$V \int_{-\infty}^{\infty} b_2 \left(\frac{1}{4} e^{\varphi} \frac{f_1^2 f_4^4}{f_2^2} \star db_2 + b_1 dj_2 \right). \qquad (3.28)$$

The sign change from (3.27) is due to the natural direction for integration. We observe that f_4 vanishes on regions of the real axis where S^4 shrinks to zero size. In fact, j_2 is constant there because otherwise F_5 that contains $dj_2 \wedge \hat{e}^{4567}$ would be ill-defined. On the other hand, b_2 is constant on regions where S^2 shrinks for a similar reason and, as we recall in appendix A, $b_2 = -4 \mathrm{Im}\, \mathcal{A}$. Since $\mathcal{A}(e_{2g+2}) = 0$ by (2.34), the flux condition (2.32) determines these constants to be

$$b_2 = 2\pi\alpha' K_I \quad \text{on} \quad [e_{2I}, e_{2I-1}]. \qquad (3.29)$$

Thus we can write (3.28) as

$$V \sum_{I=1}^{g+1} \frac{\pi}{2} \alpha' K_I \int_{[e_{2I}, e_{2I-1}]} \left(e^{\varphi} \frac{f_1^2 f_4^4}{f_2^2} \star db_2 + 4 b_1 dj_2 \right). \qquad (3.30)$$

The physical meaning of the integrand in (3.30) can be understood as follows. The equation of motion for $B_{(2)}^{\rm RR}$ can be written as

$$dH_{(7)}^{\rm RR} = 0, \qquad (3.31)$$

where

$$H_{(7)}^{\rm RR} \equiv e^{\varphi} \star H_{(3)}^{\rm RR} + 4 B_{(2)}^{\rm NS} \wedge F_{(5)} - \frac{1}{2} B_{(2)}^{\rm NS} \wedge B_{(2)}^{\rm NS} \wedge H_{(3)}^{\rm RR}. \qquad (3.32)$$

It is easy to see that the integrand in (3.30) is proportional to the component of $H_{(7)}^{\rm RR}$ along the $AdS_2 \times S^4$ direction. The seven-form is to be regarded as the field strength of the six-form potential $H_{(7)}^{\rm RR} = dC_{(6)}^{\rm RR}$. By the symmetry of $AdS_2 \times S^2 \times S^4$, we can write

$$C_{(6)}^{\rm RR} = b_4 \hat{e}^{014567}, \qquad (3.33)$$

where $\hat{e}^{014567}$ is the volume form of unit $AdS_2 \times S^4$. Then by definition

$$e^{\varphi} \frac{f_1^2 f_4^4}{f_2^2} \star db_2 + 4 b_1 dj_2 = db_4. \tag{3.34}$$

Thus the integrand in (3.30) is db_4.

One can express the l.h.s. of (3.34) in terms of $\mathcal{A}$ and z using the known expressions for fields summarized in appendix A. It is in fact possible to integrate the equation:

$$\begin{aligned} \frac{1}{\alpha'^2} b_4 = & \frac{2(z-\bar{z})(\mathcal{A}+\bar{\mathcal{A}})^2 - (z^2-\bar{z}^2)(\mathcal{A}+\bar{\mathcal{A}})(\partial_z \mathcal{A} + \partial_{\bar{z}} \bar{\mathcal{A}})}{2(\partial_z \mathcal{A} - \partial_{\bar{z}} \bar{\mathcal{A}})} \\ & + \frac{3}{2}(z^2-\bar{z}^2)(\mathcal{A}-\bar{\mathcal{A}}) - 6 \int dz\, z \mathcal{A} - 6 \int d\bar{z}\, \bar{z} \bar{\mathcal{A}}, \end{aligned} \tag{3.35}$$

where the last two terms involve indefinite integrals. One can check that (3.34) is satisfied by this solution. On the real axis where $z = \bar{z}$, b_4 reduces to

$$b_4 = -6\alpha'^2 \int dz\, z\, \mathcal{A} + c.c.\,. \tag{3.36}$$

Thus

$$b_4(e_{2I-1}) - b_4(e_{2I}) = -6\pi^2 \alpha'^3 N \int_{[e_{2I}, e_{2I-1}]} dx \rho(x) x\,. \tag{3.37}$$

By collecting everything together, (3.27) becomes

$$2\kappa^2 \mathcal{S}_{\text{bulk}}/V = -\frac{3}{2}\pi^3 \alpha'^4 N \sum_{I=1}^{g+1} K_I \int_{[e_{2I}, e_{2I-1}]} dx \rho(x) x\,. \tag{3.38}$$

This is the contribution from the bulk, in particular from the cycles that have grown in the bubbling geometry. This is not the complete story, since the volume V should be regularized and counter-terms on the boundary should be added. We see indeed that (3.38) seems to account only for special terms in the matrix model action in (3.11).

4. Conclusion

The main achievement of this paper is the large N solution of the matrix model that governs circular BPS Wilson loops at strong coupling. We determined the eigenvalue distribution for an arbitrary representation in terms of geometric data on the spectral curve. The spectral curve was then identified with the hyperelliptic surface Σ that was found in [7] to characterize the bubbling geometry for the Wilson loop.

The identification of the hyperelliptic surface Σ as a spectral curve is important for two reasons. First, one can view this as an example of emergent geometry. The matrix model is a reduction of the four-dimensional gauge theory [14] and the geometry emerges out of the dynamics of the eigenvalues.

Second, the identification provides the precise dictionary between field theory and gravity. Indeed it serves as the basis for the matching of physical quantities computed on

JHEP09(2008)050

both sides. A successful example of matching is reported in [17], where the correlators of the Wilson loop with chiral primaries and the energy-momentum tensor are computed.

It should also be possible to match the computations of the Wilson loop expectation value. Given our solution of the matrix model, we were able to compute the Wilson loop expectation value quite easily. On the other hand, the computation of the expectation value in supergravity is unfinished. Such computation should involve two non-trivial tasks. One is to properly take into account the new cycles that appear in the bubbling geometry. In the present work, we developed techniques to perform this task. The other task is to regulate the infinite volume of the ten-dimensional space-time and to add proper counter-terms. Usual five-dimensional counter-terms do not suffice, because the bubbling geometry mixes the AdS_5 and S^5 directions in a topologically non-trivial way.[7] Construction of the counter-terms is a worthwhile open problem that has applications to other observables such as surface operators [37].

Acknowledgments

We are happy to thank David Berenstein, Eric D'Hoker, Sean Hartnoll, Gary Horowitz, Don Marolf, David Mateos, Joe Polchinski, and especially Kostas Skenderis for fruitful discussions. We are also grateful to Jaume Gomis and Shunji Matsuura for collaboration on a related project. This research is supported by the NSF grants PHY05-51164 and PHY04-56556, and by the Department of Energy under Contract DE-FG02-91ER40618.

A. Details on the bubbling geometry

The solution to the BPS equations can be expressed in terms of two holomorphic functions $\mathcal{A}$ and $\mathcal{B}$ on the lower half-plane. Let us define four harmonic functions h_1, $\tilde{h}_1$, h_2, and $\tilde{h}_2$ by

$$\mathcal{A} \equiv \frac{1}{2}(h_1 - i\tilde{h}_1)\,, \qquad \mathcal{B} \equiv \frac{1}{2}(h_2 - i\tilde{h}_2). \tag{A.1}$$

In fact, all the physical fields except the form fields can be written in terms of h_1 and h_2 alone. The field strengths of the form fields are also given in terms of h_1 and h_2. The dual harmonic functions $\tilde{h}_1$ and $\tilde{h}_2$ only appear in the potentials [7].

It is useful to define the following shorthand notations

$$\begin{aligned} V &= \partial_w h_1 \partial_{\bar{w}} h_2 - \partial_{\bar{w}} h_1 \partial_w h_2\,, \qquad & W &= \partial_w h_1 \partial_{\bar{w}} h_2 + \partial_{\bar{w}} h_1 \partial_w h_2\,, \\ N_1 &= 2h_1 h_2 |\partial_w h_1|^2 - h_1^2 W\,, \qquad & N_2 &= 2h_1 h_2 |\partial_w h_2|^2 - h_2^2 W, \end{aligned} \tag{A.2}$$

where w is an arbitrary complex coordinate on Σ. Then we have

$$\begin{aligned} e^{2\varphi} &= -\frac{N_2}{N_1}\,, \qquad & \rho^8 &= -\frac{W^2 N_1 N_2}{h_1^4 h_2^4}\,, \\ f_1^4 &= -4e^{\varphi} h_1^4 \frac{W}{N_1}\,, \qquad & f_2^4 &= 4e^{-\varphi} h_2^4 \frac{W}{N_2}\,, \qquad & f_4^4 &= 4e^{-\varphi}\frac{N_2}{W}\,, \end{aligned} \tag{A.3}$$

[7] A similar problem, related to the difficulties of formulating higher-dimensional counter-terms [35], was already encountered by one of the present authors in the context of bubbling geometries [36].

while the relevant components of the two- and four-form fields (3.21) are

$$b_1 = -2i\frac{h_1^2 h_2 V}{N_1} - 2\tilde{h}_2\,, \qquad b_2 = -2i\frac{h_1 h_2^2 V}{N_2} + 2\tilde{h}_1, \tag{A.4}$$

as given in [7], and

$$j_2 = ih_1h_2\frac{V}{W} + 3i(\mathcal{C} - \bar{\mathcal{C}}) - \frac{3}{2}(\tilde{h}_1 h_2 - h_1\tilde{h}_2)\,, \tag{A.5}$$

as we show in appendix B. The holomorphic function $\mathcal{C}$ is defined implicitly by

$$\partial_w \mathcal{C} = \mathcal{A}\partial_w \mathcal{B} - \mathcal{B}\partial_w \mathcal{A}\,. \tag{A.6}$$

The behavior of various quantities near the real axis ($y = 0$) was studied in [7]:

Intervals	Vanishing fiber	h_1	$\partial_y h_1$	h_2	V	W	N_1	N_2
$[e_{2I}, e_{2I-1}]$	S^2	$\mathcal{O}(1)$	$\mathcal{O}(y)$	$\mathcal{O}(y)$	$\mathcal{O}(1)$	$\mathcal{O}(y)$	$\mathcal{O}(y)$	$\mathcal{O}(y)$
others	S^4	$\mathcal{O}(y)$	$\mathcal{O}(1)$	$\mathcal{O}(y)$	$\mathcal{O}(y)$	$\mathcal{O}(1)$	$\mathcal{O}(y^4)$	$\mathcal{O}(y^4)$

(A.7)

It follows that $b_2 = 2\tilde{h}_1 = -4\mathrm{Im}\,\mathcal{A}$ on $[e_{2I}, e_{2I-1}]$.

B. An explicit expression for the four-form

The component j_2 of the RR four-form $C_{(4)}$ (3.21) is not given explicitly in [7], but can be obtained along the lines of the similar computation in section 9.9 of [38]. We use the notation of these papers.

The derivative of j_2 admits an expression[8]

$$dj_2 = -if_4^4\,(\rho f_w dw - \rho f_{\bar{w}} d\bar{w})\,, \tag{B.1}$$

where from eqs. (5.24) and (6.1) of [7] and from the relation $\rho p_w = \partial_w \phi$ ($\phi \equiv \varphi/2$) one has

$$2\rho f_w = \partial_w \log\frac{\bar{\beta}}{\bar{\alpha}} + \left(\frac{\beta\bar{\beta}}{\alpha\bar{\alpha}} - \frac{\alpha\bar{\alpha}}{\beta\bar{\beta}}\right)\partial_w\phi. \tag{B.2}$$

Using that

$$\alpha = \sqrt{\frac{\bar{\kappa}}{\rho}}\sqrt{\cosh(\phi+\bar{\lambda})}\,, \qquad \beta = i\sqrt{\frac{\bar{\kappa}}{\rho}}\sqrt{\sinh(\phi+\bar{\lambda})}\,, \tag{B.3}$$

it becomes

$$2\rho f_w = \frac{\partial_w\phi + \partial_w\lambda}{\sinh(2\phi+2\lambda)} - \frac{2\cosh(\lambda-\bar{\lambda})}{|\sinh(2\phi+2\lambda)|}\partial_w\phi\,. \tag{B.4}$$

The warp factor is given by $f_4 = \nu(\bar{\alpha}\beta + \bar{\beta}\alpha)$ (with $\nu = \pm 1$), so that

$$f_4^4 = \frac{\kappa^2\bar{\kappa}^2}{\rho^4}\left(\sinh(2\phi+\lambda+\bar{\lambda}) - |\sinh(2\phi+2\lambda)|\right)^2\,. \tag{B.5}$$

[8]Using complex coordinates on Σ, the frames become $e^8 = e^w + e^{\bar{w}}$, $e^9 = -i(e^w - e^{\bar{w}})$, with $e^w = \rho\,dw$ and $e^{\bar{w}} = \rho\,d\bar{w}$.

One can now change the variables from ϕ (real) and λ (holomorphic) to the real variables μ and ϑ defined by

$$\lambda - \bar{\lambda} = i\mu\,, \qquad e^{2i\vartheta} = \frac{\sinh(2\phi+2\lambda)}{\sinh(2\phi+2\bar{\lambda})}\,, \tag{B.6}$$

from which also follows

$$|\sinh(2\phi+2\lambda)|^2 = \frac{(\sin 2\mu)^2}{4\sin(\vartheta+\mu)\sin(\vartheta-\mu)}\,, \qquad e^{-i\vartheta} = \frac{|\sinh(2\phi+2\lambda)|}{\sinh(2\phi+2\lambda)}\,, \tag{B.7}$$

and

$$\partial_w\phi = -\frac{\sin 2\mu\,\partial_w\vartheta}{4\sin(\vartheta+\mu)\sin(\vartheta-\mu)} - \frac{i}{2}\partial_w\mu + \frac{\sin 2\vartheta\,\partial_w\mu}{4\sin(\vartheta+\mu)\sin(\vartheta-\mu)}\,. \tag{B.8}$$

Using eq. (7.4) of [7] one has

$$\begin{aligned} 2\rho f_w f_4^4 \,=\, & \frac{1}{2\hat{\rho}^4\cos^2\mu}\Big[e^{-i\vartheta}\left(-\sin 2\mu\,\partial_w\vartheta - ie^{2i\vartheta}\,\partial_w\mu + i\cos 2\mu\,\partial_w\mu\right) - \\ & -2\cos\mu\left(-\sin 2\mu\,\partial_w\vartheta + ie^{-2i\vartheta}\,\partial_w\mu - i\cos 2\mu\,\partial_w\mu\right)\Big]\,. \end{aligned} \tag{B.9}$$

In terms of $\psi = \frac{\sin\mu}{\hat{\rho}^2}e^{-i\vartheta/2}$, the expression above becomes

$$\begin{aligned} 2\rho f_w f_4^4 \,=\, & \frac{2i}{(\sin 2\mu)^2}\Big[-\sin 2\mu\left(\psi\,\partial_w\psi - \psi^2\,\partial_w\bar{\psi}\,/\bar{\psi}\right) - \bar{\psi}^2\,\partial_w\mu + \cos 2\mu\,\psi^2\,\partial_w\mu + \\ & +2\cos\mu\sin 2\mu\left(\bar{\psi}\,\partial_w\psi - \psi\,\partial_w\bar{\psi}\right) - 2\cos\mu\,\psi^3\,\partial_w\mu\,/\bar{\psi} + \\ & +2\cos\mu\cos 2\mu\,\psi\bar{\psi}\,\partial_w\mu\Big]\,, \end{aligned} \tag{B.10}$$

and finally, using the equation of motion

$$\partial_w\bar{\psi} = \cot\mu\,\bar{\psi}\,\partial_w\mu + \frac{1}{\sin\mu}\psi\,\partial_w\mu \tag{B.11}$$

to eliminate the pieces with more than 2 ψ and/or $\bar{\psi}$,

$$\begin{aligned} 2\rho f_w f_4^4 \,=\, 2i\Big[& -\frac{\psi\,\partial_w\psi}{\sin 2\mu} + \frac{\psi^2-\bar{\psi}^2}{(\sin 2\mu)^2}\partial_w\mu + \frac{2\cos 2\mu}{(\sin 2\mu)^2}\psi^2\,\partial_w\mu + \\ & +\frac{2\cos\mu}{\sin 2\mu}(\bar{\psi}\,\partial_w\psi - \psi\,\partial_w\bar{\psi}) + \frac{2\cos\mu\,\cos 2\mu}{(\sin 2\mu)^2}\psi\bar{\psi}\,\partial_w\mu\Big]\,. \end{aligned} \tag{B.12}$$

This can be almost written as a total derivative

$$2\rho f_w f_4^4 = \partial_w\left(2i\frac{\psi\bar{\psi}}{\sin\mu} - i\frac{\psi^2+\bar{\psi}^2}{\sin 2\mu}\right) - 3i\frac{\psi^2\,\partial_w\mu}{\sin^2\mu}\,, \tag{B.13}$$

using again the equation of motion for $\bar{\psi}$. Using the equation of motion for ψ, eq. (7.7) of [7], the expression for κ in eqs. (7.8) and (7.13) and the last equation in (7.14), the last term in the formula above becomes

$$-\frac{\psi^2\,\partial_w\mu}{\sin^2\mu} = \partial_w\left(\psi^2\cot\mu + ih_1^2e^{-2\bar{\lambda}} - ih_2^2e^{2\bar{\lambda}}\right) + 2i(h_1\partial_w h_2 - h_2\partial_w h_1)\,. \tag{B.14}$$

Then

$$\rho f_w f_4^4 = -i\partial_w(h_1 h_2 \tan\mu) - 3(h_1\partial_w h_2 - h_2\partial_w h_1)\,, \tag{B.15}$$

and one has

$$j_2 = -h_1 h_2 \tan\mu + 3i(\mathcal{C} - \bar{\mathcal{C}}) - \frac{3}{2}(\tilde{h}_1 h_2 - h_1 \tilde{h}_2)\,. \tag{B.16}$$

Using (A.6) together with the relations $\mu = -i(\lambda - \bar{\lambda})$ and $e^{2\lambda} = \partial_w h_1/\partial_w h_2$, one can rewrite this as (A.5).

C. Asymptotic behavior

Let us study the asymptotic forms of physical fields in the region $z \to \infty$. We use the SU(N) identification (2.42) of the matrix model and geometry data.

From the definition (2.6), ω behaves in the asymptotic region of Σ as

$$\omega(z) = \frac{\lambda}{z} + \mathcal{O}\left(\frac{1}{z^3}\right). \tag{C.1}$$

The order $\mathcal{O}(z^{-2})$ term vanishes in the SU(N) case. Using the formulas in appendix A, we find the asymptotic forms of various fields:

$$e^{\varphi} \equiv e^{2\phi} = g_s + \mathcal{O}(r^{-4}), \tag{C.2}$$

$$f_1 = \left(\frac{\alpha'^2}{g_s\lambda}\right)^{1/4} r + \mathcal{O}(1/r), \qquad f_2 = \left(\frac{\alpha'^2}{g_s\lambda}\right)^{1/4} r + \mathcal{O}(1/r), \tag{C.3}$$

$$f_4 = \left(\frac{\alpha'^2\lambda}{g_s}\right)^{1/4} |\sin\theta| + \mathcal{O}(1/r^2), \tag{C.4}$$

$$b_1 = \mathcal{O}(1/r), \qquad b_2 = \mathcal{O}(1/r), \tag{C.5}$$

$$j_2 = -\frac{\alpha'^2\lambda[12\theta - 8\sin(2\theta) + \sin(4\theta)]}{32 g_s} + \mathcal{O}(1/r^2), \tag{C.6}$$

$$b_4 = \mathcal{O}(1/r). \tag{C.7}$$

Here we introduced polar coordinates $z = re^{i\theta}$ with $-\pi \leq \theta \leq 0$. Note that the metric (1.1) is written in the Einstein frame where the AdS radius is $(\alpha'^2\lambda/g_s)^{1/4}$ in our convention (C.2) for the dilaton. The subleading terms depend on the representation R and can be easily calculated in terms of the moments of the eigenvalue distribution.

References

[1] D. Berenstein, *Large-N BPS states and emergent quantum gravity*, *JHEP* **01** (2006) 125 [hep-th/0507203].

[2] D. Berenstein, *A strong coupling expansion for N = 4 SYM theory and other SCFT's*, *Int. J. Mod. Phys.* **A 23** (2008) 2143 [arXiv:0804.0383].

[3] D. Berenstein, *A toy model for the AdS/CFT correspondence*, *JHEP* **07** (2004) 018 [hep-th/0403110].

[4] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and* 1/2 *BPS geometries*, *JHEP* **10** (2004) 025 [hep-th/0409174].

[5] S. Yamaguchi, *Bubbling geometries for half BPS Wilson lines*, *Int. J. Mod. Phys.* **A 22** (2007) 1353 [hep-th/0601089].

[6] O. Lunin, *On gravitational description of Wilson lines*, *JHEP* **06** (2006) 026 [hep-th/0604133].

[7] E. D'Hoker, J. Estes and M. Gutperle, *Gravity duals of half-BPS Wilson loops*, *JHEP* **06** (2007) 063 [arXiv:0705.1004].

[8] J. Gomis and S. Matsuura, *Bubbling surface operators and S-duality*, *JHEP* **06** (2007) 025 [arXiv:0704.1657].

[9] J. Gomis and T. Okuda, *D-branes as a bubbling Calabi-Yau*, *JHEP* **07** (2007) 005 [arXiv:0704.3080].

[10] J. Gomis and T. Okuda, *Wilson loops, geometric transitions and bubbling Calabi-Yau's*, *JHEP* **02** (2007) 083 [hep-th/0612190].

[11] N. Halmagyi and T. Okuda, *Bubbling Calabi-Yau geometry from matrix models*, *JHEP* **03** (2008) 028 [arXiv:0711.1870].

[12] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 582** (2000) 155 [hep-th/0003055].

[13] N. Drukker and D.J. Gross, *An exact prediction of $N = 4$ SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [hep-th/0010274].

[14] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, arXiv:0712.2824.

[15] G. Bonelli and H. Safaai, *On gauge/string correspondence and mirror symmetry*, *JHEP* **06** (2008) 050 [arXiv:0804.2629].

[16] N. Berkovits and C. Vafa, *Towards a worldsheet derivation of the Maldacena conjecture*, *JHEP* **03** (2008) 031 [arXiv:0711.1799].

[17] J. Gomis, S. Matsuura, T. Okuda and D. Trancanelli, *Wilson loop correlators at strong coupling: from matrices to bubbling geometries*, *JHEP* **08** (2008) 068 [arXiv:0807.3330].

[18] T. Okuda, *A prediction for bubbling geometries*, *JHEP* **01** (2008) 003 [arXiv:0708.3393].

[19] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, *JHEP* **02** (2005) 010 [hep-th/0501109].

[20] S. Yamaguchi, *Wilson loops of anti-symmetric representation and D5-branes*, *JHEP* **05** (2006) 037 [hep-th/0603208].

[21] K. Okuyama and G.W. Semenoff, *Wilson loops in $N = 4$ SYM and fermion droplets*, *JHEP* **06** (2006) 057 [hep-th/0604209].

[22] S.A. Hartnoll and S.P. Kumar, *Higher rank Wilson loops from a matrix model*, *JHEP* **08** (2006) 026 [hep-th/0605027].

JHEP09(2008)050

[23] S. Giombi, R. Ricci and D. Trancanelli, *Operator product expansion of higher rank Wilson loops from D-branes and matrix models*, *JHEP* **10** (2006) 045 [hep-th/0608077].

[24] N. Drukker, 1/4 *BPS circular loops, unstable world-sheet instantons and the matrix model*, *JHEP* **09** (2006) 004 [hep-th/0605151].

[25] N. Drukker, S. Giombi, R. Ricci and D. Trancanelli, *On the D3-brane description of some* 1/4 *BPS Wilson loops*, *JHEP* **04** (2007) 008 [hep-th/0612168].

[26] N. Drukker, S. Giombi, R. Ricci and D. Trancanelli, *More supersymmetric Wilson loops*, *Phys. Rev.* **D 76** (2007) 107703 [arXiv:0704.2237].

[27] N. Drukker, S. Giombi, R. Ricci and D. Trancanelli, *Wilson loops: from four-dimensional SYM to two-dimensional YM*, *Phys. Rev.* **D 77** (2008) 047901 [arXiv:0707.2699].

[28] N. Drukker, S. Giombi, R. Ricci and D. Trancanelli, *Supersymmetric Wilson loops on* S^3, *JHEP* **05** (2008) 017 [arXiv:0711.3226].

[29] P.H. Ginsparg and G.W. Moore, *Lectures on* $2-D$ *gravity and* $2-D$ *string theory*, hep-th/9304011.

[30] P. Di Francesco, P.H. Ginsparg and J. Zinn-Justin, $2-D$ *gravity and random matrices*, *Phys. Rept.* **254** (1995) 1 [hep-th/9306153].

[31] J. Gomis and F. Passerini, *Holographic Wilson loops*, *JHEP* **08** (2006) 074 [hep-th/0604007].

[32] J. Gomis and F. Passerini, *Wilson loops as D3-branes*, *JHEP* **01** (2007) 097 [hep-th/0612022].

[33] G. Dall'Agata, K. Lechner and D.P. Sorokin, *Covariant actions for the bosonic sector of* $D=10$ *IIB supergravity*, *Class. and Quant. Grav.* **14** (1997) L195 [hep-th/9707044].

[34] E. D'Hoker, J. Estes and M. Gutperle, *Ten-dimensional supersymmetric Janus solutions*, *Nucl. Phys.* **B 757** (2006) 79 [hep-th/0603012].

[35] M. Taylor, *Higher-dimensional formulation of counterterms*, hep-th/0110142.

[36] S. Giombi, M. Kulaxizi, R. Ricci and D. Trancanelli, *Half-BPS geometries and thermodynamics of free fermions*, *JHEP* **01** (2007) 067 [hep-th/0512101].

[37] N. Drukker, J. Gomis and S. Matsuura, *Probing* $N=4$ *SYM with surface operators*, arXiv:0805.4199.

[38] E. D'Hoker, J. Estes and M. Gutperle, *Exact half-BPS type IIB interface solutions I: local solution and supersymmetric Janus*, *JHEP* **06** (2007) 021 [arXiv:0705.0022].

JHEP09(2008)050